RATIONAL CUBIC TRIGONOMETRIC EXPONENTIAL BÉZIER CURVE

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Abstract
The cubic trigonometric basis functions are constructed by including two exponential functions with five shape parameters and a tension parameter. Also, the properties of the basis functions are discussed. We propose to define rational cubic trigonometric exponential Bézier curve with weight \( \omega \) and the continuity condition of the curve. Visual effects of parameters and tension parameter along with weight \( \omega \) are presented. We compare the proposed method of constructing the desired curve with existing method. The conic sections of the curve are also discussed.

Key words: Cubic trigonometric exponential basis function, Rational cubic trigonometric exponential Bézier curve, Shape parameter, Tension parameter, Exponential function.

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1. Introduction

In the field of Geometric Modelling and related applications, the shape of the curves usually require a free adjustment. In order to construct a curve that can be adjusted freely, the curves of free parameters are studied by many researchers [3, 6, 7, 9]. The shapes of curves can be adjusted by altering the values of free parameters. However, it might just have to construct the fairest curves in some cases. Therefore, a natural idea is that we can find the optimal values of the free parameters of curves so that the curves as fair as possible. Recently, trigonometric polynomial received much attention with in geometric modelling for example Han [7] presented a class of cubic trigonometric polynomial curve with a shape parameter, Bashir [2] (see also Bashir [3]) presented the rational quadratic and cubic trigonometric Bézier curve with two shape parameters. Yan [10] represented an algebraic trigonometric blended piecewise curve with two shape parameters, Dube and Yadav [6] presented a shape analysis of quantic trigonometric Bézier curve with two shape parameters. Exponential splines have attracted widespread interest in curve design and other field such as shape preserving interpolation and / or approximation due to them possess tension properties [1, 5, 8].

The Rational Cubic Trigonometric Exponential Bézier Curve (RCTEBC) with five shape parameters is considered, which is an extension of idea given by Beibei et al. [4]. By using above mentioned idea, the geometric properties, continuity conditions have presented. Shape control of curves are also discussed for the different values of shape parameters. The shape parameters have a predicable adjusting role on the curves.

The paper is organised as follows: Section 2 gives the definition and properties of cubic trigonometric Bézier basis functions with five shape parameter and a tension parameter. Section 3 constructs the rational cubic trigonometric Bézier curve. Section 4 discusses continuity condition for shape control. Section 5 presents shape parameter and weight effect of the shape of the curve with different examples. Section 6 compares with existing method. Section 7 represents conic sections. Section 8 concludes the paper.

2. Cubic Trigonometric Exponential Bézier Basis Function

In this section, cubic trigonometric basis function including two exponential functions with five shape parameters and with a tension parameter are given.

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**Definition 2.1:** Let $\beta$ be the tension parameter, $t \in [0, \frac{\pi}{2\beta}]$ and for $l, m, n \in [1, -1], \lambda, \mu \in [0, +\infty)$ the cubic trigonometric Bézier basis function are define as (Beibei et al. [4]):

$$
\begin{align*}
    b_{0,\beta}(t) &= (1 - \sin \beta t)(1 - l \sin \beta t)(1 - m \sin \beta t)e^{-\lambda t} \\
    b_{1,\beta}(t) &= 1 - b_{0,\beta}(t) - b_{2,\beta}(t) \\
    b_{2,\beta}(t) &= (1 - \cos \beta t)(1 - l \cos \beta t)(1 - n \cos \beta t)e^{-\mu(\frac{\pi}{2\beta} - t)}
\end{align*}
$$

(2.1)

**Theorem 2.1:** The basis function (2.1) have the following properties:

(a) **Non negativity:** $b_{i,\beta}(t) \geq 0$, $i = 0, 1, 2$.

(b) **Partition of unity:** $\sum_{i=0}^{2} b_{i,\beta}(t) = 1$.

(c) **Symmetry:** $b_{i,\beta}(t, l, m, n, \lambda, \mu) = b_{2-i}(\frac{\pi}{2\beta} - t; l, m, n, \lambda, \mu)$, $i = 0, 1, 2$.

(d) **Monotonicity:** $b_{0,\beta}(t)$ is monotonically decreasing for fixed $t \in [0, +\infty]$ and for shape parameters $l, m$ and $\lambda$. $b_{2,\beta}(t)$ is monotonically decreasing for shape parameters $l, m, n, \lambda$ and $\mu$. 

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(a) **Properties of end points:**

\[ b_{0,\beta}(0) = 1, \quad b_{1,\beta}(0) = 0, \quad b_{2,\beta}(0) = 0; \]

\[ b_{0,\beta}\left(\frac{\pi}{2\beta}\right) = 0, \quad b_{1,\beta}\left(\frac{\pi}{2\beta}\right) = 0, \quad b_{2,\beta}\left(\frac{\pi}{2\beta}\right) = 1; \]

\[ b'_{0,\beta}(0) = -[\lambda + (1 + m + l)\beta], \quad b'_{1,\beta}(0) = [\lambda + (1 + m + l)\beta], \quad b'_{2,\beta}(0) = 0, \]

\[ b'_{0,\beta}\left(\frac{\pi}{2\beta}\right) = 0, \quad b'_{1,\beta}\left(\frac{\pi}{2\beta}\right) = -[\mu + (1 + n + l)\beta], \quad b'_{2,\beta}\left(\frac{\pi}{2\beta}\right) = [\mu + (1 + n + l)\beta], \]

\[ b''_{0,\beta}(0) = \lambda^2 + 2\lambda\beta(1 + m + l) + \beta^2(2m + 2l + 2lm), \]

\[ b''_{1,\beta}(0) = -\lambda^2 - 2\lambda\beta(1 + m + l) - \beta^2(2m + 2l + 2lm) - \beta^2e^{-\frac{\mu\pi}{2\beta}}(1 - l)(1 - n), \]

\[ b''_{2,\beta}(0) = \beta^2e^{-\frac{\lambda\pi}{2\beta}}(1 - l)(1 - m), \]

\[ b''_{0,\beta}\left(\frac{\pi}{2\beta}\right) = \beta^2e^{-\frac{\lambda\pi}{2\beta}}(1 - l)(1 - m) \]

\[ b''_{1,\beta}\left(\frac{\pi}{2\beta}\right) = -\mu^2 - 2\mu\beta(1 + n + l) - \beta^2(2n + 2l + 2ln) - \beta^2e^{-\frac{\lambda\pi}{2\beta}}(1 - l)(1 - m), \]

\[ b''_{2,\beta}\left(\frac{\pi}{2\beta}\right) = \mu^2 + 2\mu\beta(1 + n + l) + \beta^2(2n + 2l + 2ln). \]

**Proof:**

(a) For \( l, m, n \in [-1, 1] \) and \( \lambda, \mu \in [0, +\infty) \), then

\[
(1 - \sin \beta t) \geq 0, \quad (1 - l \sin \beta t) \geq 0, \quad (1 - m \sin \beta t) \geq 0
\]

\[
(1 - \cos \beta t) \geq 0, \quad (1 - l \cos \beta t) \geq 0, \quad (1 - n \cos \beta t) \geq 0
\]

It is obvious that

\[ b_{i,\beta}(t) \geq 0, \quad i = 0, 1, 2. \]

(b) \( \sum_{i=0}^{2} b_{i,\beta}(t) = (1 - \sin \beta t)(1 - l \sin \beta t)(1 - m \sin \beta t)e^{-\lambda t} + \)

\[
1 - (1 - \sin \beta t)(1 - l \sin \beta t)(1 - m \sin \beta t)e^{-\lambda t} + (1 - \cos \beta t)(1 - l \cos \beta t)(1 - n \cos \beta t)e^{-\mu(\pi/2\beta-t)}
\]

\[ + (1 - \cos \beta t)(1 - l \cos \beta t)(1 - n \cos \beta t)e^{-\mu(\pi/2\beta-t)} = 1. \]

(c) For \( i = 2, \)

\[ b_{2,\beta}(t, l, m, n, \lambda, \mu) \]

\[ = (1 - \cos \beta t)(1 - l \cos \beta t)(1 - n \cos \beta t)e^{-\mu(\pi/2\beta-t)} \]

\[ = (1 - \cos \beta \left(\frac{\pi}{2\beta} - t\right))(1 - l \cos \beta \left(\frac{\pi}{2\beta} - t\right))(1 - n \cos \beta \left(\frac{\pi}{2\beta} - t\right))e^{-\mu(\pi/2\beta-t)} \]

\[ = (1 - \cos \left(\frac{\pi}{2} - \beta t\right))(1 - l \cos \left(\frac{\pi}{2} - \beta t\right))(1 - n \cos \left(\frac{\pi}{2} - \beta t\right))e^{-\mu t} \]

\[ = (1 - \sin \beta t)(1 - l \sin \beta t)(1 - n \sin \beta t)e^{-\mu t} \]

\[ = b_{0,\beta}\left(\frac{\pi}{2\beta} - t; l, m, n, \lambda, \mu\right). \]
Fig 1: Basis function of RCTEBC

(d) Monotonicity of function has been shown in Fig. 1. The curves of the cubic trigonometric Bézier basis functions for $\lambda = \mu = l = m = n = 0$ (dotted lines), for $\lambda = \mu = 1$, $l = m = n = 0$ (solid lines), for $\lambda = \mu = l = m = n = 1$ (dashes lines) with shape parameter $\beta = \frac{\pi}{2}$ have been shown in Fig. 1.

3. The Rational Cubic Trigonometric Exponential Bézier Curve

Definition 3.1: The Rational cubic trigonometric exponential Bézier curve with five parameter and with tension parameter is defined as:

$$S(t) = \frac{b_{0,\beta}(t)V_0 + b_{1,\beta}(t)V_1\omega + b_{2,\beta}(t)V_2}{b_{0,\beta}(t) + b_{1,\beta}(t)\omega + b_{2,\beta}(t)}, \quad t \in \left[0, \frac{\pi}{2\beta}\right] \quad (3.1)$$

Let $V_i(i = 0,1,2)$ are three control points in $R^d(d = 2,3), b_{i,\beta}(i = 0,1,2)$ are the basis function defined in (2.1) where $\omega > 0$ is called weight of the function.

Theorem 3.1: The Rational cubic trigonometric exponential Bézier curve (3.1) have the following properties:

(a) End point properties:

$$S(0) = V_0,$$

$$S\left(\frac{\pi}{2\beta}\right) = V_2$$

$$S'(0) = \omega[\lambda + \beta(1 + m + l)](V_1 - V_0),$$

$$S'(\frac{\pi}{2\beta}) = \omega[\mu + \beta(1 + n + l)](V_2 - V_1)$$

$$S''(0) = [\lambda^2\omega + 2\lambda\beta(1 + m + l)\omega - \beta^2\omega[2m + 2l + 2lm]$$

$$-\beta^2e^{-\frac{\mu}{2\beta}}(1-l)(1-n)\omega + 2\beta^2(1 + m + l)\omega - 2\lambda^2\omega^2$$

$$-4\lambda\beta(1 + m + l)\omega^2 - 2\beta^2(1 + m + l)\omega^2](V_1 - V_0)$$

$$+\beta^2e^{-\frac{\mu}{2\beta}}[(1-l)(1-n)(V_2 - V_0).$$

$$S''\left(\frac{\pi}{2\beta}\right) = [-\mu^2\omega - 2\mu\beta(1 + n + l)\omega + \beta^2[2n + 2l + 2ln]\omega$$

$$+\beta^2e^{-\frac{\mu}{2\beta}}(1-l)(1-n)(V_2 - V_0).$$

$$S''\left(\frac{\pi}{2\beta}\right) = [-\mu^2\omega - 2\mu\beta(1 + n + l)\omega + \beta^2[2n + 2l + 2ln]\omega$$

$$+\beta^2e^{-\frac{\mu}{2\beta}}(1-l)(1-n)(V_2 - V_0).$$
(b) **Geometric invariance:** The shape of the rational cubic trigonometric Bézier curve is independent of the choice of coordinates, i.e.
\[
S(t; V_0 + q, V_1 + q, V_2 + q) = S(t; V_0, V_1, V_2) + q,
\]
\[
S(t; V_0 * T, V_1 * T, V_2 * T) = S(t; V_0, V_1, V_2) * T,
\]
Where \( q \) is an arbitrary vector in \( R^2 \) or \( R^3 \) and \( T \) is an arbitrary \( n \times n \) matrix \( n = 2 \) or \( 3 \).

(c) **Symmetry:** \( V_0, V_1, V_2 \) and \( V_2, V_1, V_0 \) define the same RCTBC in different parameterization if the weight \( \omega \) is kept fixed, i.e.
\[
S(t; l, m, n, \lambda, \mu, V_0, V_1, V_2) = S\left(\frac{\pi}{2\beta} - t; l, m, n, \lambda, \mu, V_2, V_1, V_0\right),
\]
where, \( t \in [0, \frac{\pi}{2\beta}] \), \( l, m, n \in [1, -1] \) and \( \lambda, \mu \in [0, +\infty) \).

(d) **Convex hull property:** From partition of unity of basis function and from the non-negativity, it follows that the whole curve must lie inside its control polygon spanned by \( V_0, V_1, V_2 \).

**Remark 1:** If \( \omega = 1 \) the curve (3.1.1) will be called a cubic trigonometric Bézier curve with five shape parameters. If \( \lambda = \mu = 0 \). The rational cubic trigonometric Bézier curve will reduce to the rational cubic trigonometric Bézier curve with shape parameters.

### 4. Continuity of the Curves

Let RCTEB \( S(t) \) which is given in (3.1) and second RCTEB \( S^*(t) \) be defined by
\[
S^*(t) = \frac{b_{0, \beta}(t)V_0^* + b_{1, \beta}(t)V_1^* \omega^* + b_{2, \beta}(t)V_2^*}{b_{0, \beta}(t) + b_{1, \beta}(t)\omega^* + b_{2, \beta}(t)};
\]
where weight \( \omega^* > 0 \), shape parameters \( l^*, m^*, n^* \in [1, -1] \).

**Theorem 4.1:** Given two segments of RCTEB curves \( S(t) \) and \( S^*(t) \), then the necessary and sufficient condition of continuity is

(i) For \( C^0 \) continuity,
\[
V_0^* = V_2^*;
\]
(ii) For \( C^1 \) continuity,
\[
(V_1^* - V_0^*) = \frac{\omega[\mu + \beta(1 + n + l)]}{\omega^*[\lambda^* + \beta^*(1 + m^* + l^*)]} (V_2 - V_1)
\]
(iii) For \( C^2 \) continuity,
\[
V_2^* = V_0 + [2\omega^2 - 2\omega + 2\omega \omega^*](V_2 - V_1)
\]

**Proof:** for \( C^0 \) continuity from equation (4.1.1) we have
\[
S^*(0) = \frac{b_0(0)V_0^* + b_1(0)V_1^* \omega^* + b_2(0)V_2^*}{b_0(0) + b_1(0)\omega^* + b_2(0)} = V_0^*
\]
\[
S^*(\frac{\pi}{2\beta}) = \frac{b_0\left(\frac{\pi}{2\beta}\right)V_0 + b_1\left(\frac{\pi}{2\beta}\right)V_1 \omega + b_2\left(\frac{\pi}{2\beta}\right)V_2}{b_0\left(\frac{\pi}{2\beta}\right) + b_1\left(\frac{\pi}{2\beta}\right)\omega + b_2\left(\frac{\pi}{2\beta}\right)} = V_2
\]
\[
S^*(0) = S\left(\frac{\pi}{2\beta}\right)
\]
\[
V_0^* = V_2
\]
For $C^1$ continuity, the tangent of the two curves at the joint must be equal, that is,

$$V_0^* = V_2$$

$$S''(0) = S'(\pi/2\beta)$$

Then,

$$\omega^*[\lambda^* + \beta^*(1 + m^* + l^*)](V_1^* - V_0^*) = \omega[\mu + \beta(1 + n + l)](V_2 - V_1)$$

$$S''(0) = \omega^*[\lambda^* + \beta^*(1 + m^* + l^*)] (V_2 - V_1)$$

The two curves are joined by $C^2$ continuity if

$$V_0^* = V_2$$

$$S''(0) = S'(\pi/2\beta)$$

$$S''''(0) = S''''(\pi/2\beta)$$

Taking $l = m = n = \lambda = \mu = l^* = m^* = n^* = \lambda^* = \mu^* = 0$ and $\beta = 1$

From equation (3.2) we have

$$S''(\pi/2\beta) = (V_0 - V_2) + (2\omega^2 - \omega)(V_2 - V_1)$$

$$S''''(0) = (V_2^* - V_0^*) + (\omega^* - 2\omega^{*2})(V_1^* - V_0^*)$$

$$S''''(0) = S''''(\pi/2\beta)$$

Since $V_0^* = V_2$, $\omega^*(V_1^* - V_0^*) = \omega(V_2 - V_1)$ and $\omega^* = (1 - \omega)$

Then condition (iii) follows.

5. Effects of Parameters on the Shape of RCTEBC

By altering the values of shape parameters $l, m, n$ exponential parameters $\lambda, \mu$ and tension parameter $\beta$ for given control points, can adjust the shape of RCTEBC curve, weight $\omega$ also offers an additional control on the curve.

Fig 2: Effect on the shape of the curve with different values of parameter $l$

Fig 3: Effect on the shape of the curve with different values of parameter $m$
Fig 4: Effect on the shape of the curve with different values of parameter $n$

Fig 5: Effect on the shape of the curve with different values of parameter $\omega$

Fig 6: Effect on the shape of the curve with different values of parameter $\lambda$

Fig 7: Effect on the shape of the curve with different values of parameter $\mu$

Fig 8: Effect on the shape of the curve with different values of parameter $\beta$

Fig 9: Effect on the shape of the curve with different values of parameter $\beta$
In figure 2 curves are generated by setting the values of $\beta = \frac{\pi}{2}, \lambda = \frac{1}{2}, \mu = \frac{1}{2}, \omega = 2, l = 1, m = 0, n = 0$ (red line), $l = 0, m = 0, n = 0$ (green line) and $l = -\frac{1}{2}, m = 0, n = 0$ (pink line).

In figure 3 curves are generated for the values of shape parameters $\beta = \frac{\pi}{2}, \lambda = \frac{1}{2}, \mu = \frac{1}{2}, \omega = 2, l = 0, m = 1, n = 0$ (red line), $l = 0, m = 0, n = 0$ (green line) and $l = 0, m = -\frac{1}{2}, n = 0$ (pink line).

In figure 4 curves are generated for the values of shape parameters $\beta = \frac{\pi}{2}, \lambda = \frac{1}{2}, \mu = \frac{1}{2}, \omega = 2, l = 0, m = 0, n = 1$ (red line), $l = 0, m = 0, n = 0$ (green line) and $l = 0, m = 0, n = -\frac{1}{2}$ (pink line).

In figure 5 curves are generated for the values of shape parameters $\beta = \frac{\pi}{2}, \lambda = \frac{1}{2}, \mu = \frac{1}{2}, \omega = 3, l = 0, m = 0, n = 0$ (red line), $\omega = 2, l = 0, m = 0, n = 0$ (green line) and $\omega = 1, l = 0, m = 0, n = 0$ (pink line), respectively.

In figure 6 curves are generated for the values of shape parameters $\beta = \frac{\pi}{2}, \lambda = 0, \omega = 2, l = 0, m = 0, n = 0$ (red line), $\lambda = 3$ (green line) and $\lambda = 1$ (pink line), respectively.

In figure 7 curves are generated for the values of shape parameters $\beta = \frac{\pi}{2}, \lambda = 0, \omega = 2, l = 0, m = 0, n = 0$ (red line), $\mu = 3$ (green line) and $\mu = 1$ (pink line), respectively.

In figure 8 curves are generated for the values of shape parameters $\lambda = \frac{1}{2}, \mu = \frac{1}{2}, \omega = 2, l = \frac{1}{2}, m = -1/2, n = 1, \beta = \frac{\pi}{6}$ (red line), $\beta = \frac{\pi}{4}$ (green line) and $\beta = \frac{\pi}{2}$ (pink line).

In figure 9 curves are generated for the values of shape parameters $\beta = \frac{\pi}{2}, \lambda = \frac{1}{2}, \mu = \frac{1}{2}, \omega = 2, l = 1, m = -1/2, n = 1/2$ (red line), $l = \frac{1}{2}, m = 1, n = -\frac{1}{2}$ (green line) and $l = -\frac{1}{2}, m = \frac{1}{2}, n = 1$ (pink line).

In figure 10(a), 10(b) the closed rational cubic trigonometric Bézier curve of altering the values of the shape parameters $l, m, n, \lambda, \mu$ at the same time.

Fig 10(a): Closed rational cubic trigonometric exponential Bézier curves

Fig 10(b): Closed rational cubic trigonometric exponential Bézier curves
6. Comparison with Existing Method

In this section, proposed method is compared with existing method Beibei et al. [4]. For construction of required curve, smoothness is a very important property of the curve, so that comparison has been done with respect to smoothness of the curve. Red curve and blue curve in Figure 11 show the curves of proposed method and curve of Beibei et al. [4] respectively. We can conclude easily that curve of proposed method is more smooth than the curve of Beibei et al. [4].

![Fig 11(a): Comparison with Beibei et al. 2017 for open curve](image)

![Fig 11(b): Comparison with Beibei et al. 2017 for closed curve](image)

![Fig 11(c): Comparison with Beibei et al. 2017 for closed curve](image)

7. Conic Sections

By choosing the appropriate control points with appropriate shape parameters, weights, and special exponential parameters, the resulting RCTEBC can be used to draw some special curves like circle, parabola, ellipse and line segments. Figure 12 represent the arc of a circle, the arc of a parabola and the arc of an ellipse.
8. Conclusion

In this paper, we have presented the rational cubic trigonometric Bézier curve with exponential function and five shape parameters. It has been analysed that the behaviour of RCTEBC and ordinary cubic Bézier curve are similar. Many examples are given to design different shapes of the curve by changing the values of parameters without changing control polygon. Moreover, weight and tension parameter of RCTEBC also provide shape control of curves as required. We have compared the proposed method with method of Beibei et al. [4]. It is clear from the Figure 11 that proposed method gives smoother curve as compared to other existing method. Therefore, the proposed method is more suitable to draw curve which may improve the quality and accuracy of the graphics.

References


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